Do Covariant Projection Elements Really Satisfy the Inclusion Condition?

S. Caorsi, P. Fernandes, and M. Raffetto

***Abstract*—Spurious modes which often are found among finite-element solutions of electromagnetic eigenvalue problems do not occur when covariant projection elements are used. It has been claimed that this happens because covariant projection elements satisfy the inclusion condition, but they do not satisfy it—as is proven in this paper.**

I. Introduction

It is well known that depending on the type of elements used, finite-element solutions of electromagnetic eigenproblems may or may not contain spurious modes. For example, the spectrum calculated by using Lagrangian elements is usually polluted by spurious modes [1]–[5], whereas it is well known by experience [5], [6], and has rigorously been proved [7], [8] that edge element approximations are spectrally correct (and, in particular, are spurious free).

In [9], Crowley, Silvester, and Hurwitz introduced the covariant projection elements and proposed the so-called inclusion condition as a sufficient condition to avoid spurious modes. However, we have recently proved [10], [11] that the inclusion condition need not be satisfied by a spurious-free finite-element basis. In [10], we point out also that in practice, the inclusion condition seems too strong to be useful. As a matter of fact, edge elements (though spurious free) do not satisfy the inclusion condition [10], [11].

It is still an open question whether covariant projection elements, which are known by experience as spurious free, satisfy the inclusion condition or not. In [9] it is claimed they do, but no proof is reported. In this paper, it is proven by providing a counter example that they do not.

II. A Simple Example not Satisfying the Inclusion Condition

Let us consider the problem of finding the eigenmodes at the cutoff of the square waveguide whose cross section is the domain

$$\Omega = \{(x, y) \in R^2 | -1 \le x \le 1, -1 \le y \le 1\}. \tag{1}$$

Assuming that

$$\varepsilon_r = \mu_r = 1 \quad \text{in } \Omega \tag{2}$$

this problem reads

$$\nabla \times (\nabla \times \boldsymbol{E}) = k^2 \boldsymbol{E} \quad \text{in } \Omega \tag{3}$$

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \quad \text{on } \Omega \tag{4}$$

or, in weak form:

Find $k^2 \ge 0$ and $\boldsymbol{E} \in H_0(\text{curl}, \Omega), \boldsymbol{E} \ne 0$ such that

$$\int_\Omega (\nabla \times \boldsymbol{E}) \cdot (\nabla \times \boldsymbol{W})\, d\Omega = k^2 \int_\Omega \boldsymbol{E} \cdot \boldsymbol{W}\, d\Omega, \quad \forall \boldsymbol{W} \in H_0(\text{curl}, \Omega) \tag{5}$$

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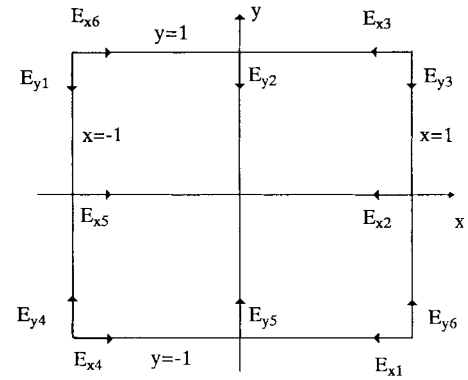

Fig. 1. The cross section of a square waveguide discretized by using a single covariant projection element.

where

$$H_0(\text{curl}, \Omega) = \{\boldsymbol{v} \in [L^2(\omega)]^3 | \nabla \times \boldsymbol{v} \in [L^2(\Omega)]^3, \boldsymbol{n} \times \boldsymbol{E}|_{\partial\Omega} = 0\} \tag{6}$$

and

$$[L^2(\Omega)]^3 = \left\{ \boldsymbol{v}: \Omega \to R^3 \middle| \int_\Omega \boldsymbol{v} \cdot \boldsymbol{v}\, d\Omega < \infty \right\}. \tag{7}$$

Let us discretize the above eigenvalue problem by using a single covariant projection element [9], [12] (see Fig. 1). The electric field is then interpolated as follows [12]:

$$\boldsymbol{E} = \sum_{i=1}^{6} E_{xi} X_i(x, y) \hat{\boldsymbol{x}} + \sum_{i=1}^{6} E_{yi} Y_i(x, y) \hat{\boldsymbol{y}} \tag{8}$$

where $X_i(x, y)$ and $Y_i(x, y), i = 1, \cdots, 6$ are the following mixed-order trial functions:

$$X_i(x, y) = -l_2(x) q_i(y), \qquad i = 1, 2, 3 \tag{9}$$

$$X_i(x, y) = l_1(x) q_{i-3}(y), \qquad i = 4, 5, 6 \tag{10}$$

$$Y_i(x, y) = X_i(y, x), \qquad i = 1, \cdots, 6 \tag{11}$$

and

$$l_1(x) = \tfrac{1}{2}(1 - x), \qquad l_2(x) = \tfrac{1}{2}(1 + x), \tag{12}$$

$$q_1(y) = \tfrac{1}{2} y(y - 1), \qquad q_2(y) = (1 - y^2), \qquad q_3(y) = \tfrac{1}{2} y(y + 1). \tag{13}$$

In order to satisfy (4), we have to choose $E_{x1} = E_{x2} = E_{x4} = E_{x6} = 0$ and $E_{y1} = E_{y3} = E_{y4} = E_{y6} = 0$, i.e., the only degrees of freedom are E_{x2}, E_{x5}, E_{y2}, and E_{y5} (see Fig. 1).

Among the possible finite-element fields satisfying (4), let us consider the following ones:

$$\boldsymbol{E}_1 = (X_2(x, y) + X_5(x, y))\hat{\boldsymbol{x}} + (Y_2(x, y) + Y_5(x, y))\hat{\boldsymbol{y}} = (x(y^2 - 1))\hat{\boldsymbol{x}} + (y(x^2 - 1))\hat{\boldsymbol{y}}, \tag{14}$$

$$\boldsymbol{E}_2 = (X_2(x, y) - X_5(x, y))\hat{\boldsymbol{x}} = (y^2 - 1)\hat{\boldsymbol{x}}, \tag{15}$$

$$\boldsymbol{E}_3 = (Y_2(x, y) - Y_5(x, y))\hat{\boldsymbol{y}} = (x^2 - 1)\hat{\boldsymbol{y}}, \tag{16}$$

$$\boldsymbol{E}_4 = (X_2(x, y) + X_5(x, y))\hat{\boldsymbol{x}} - (Y_2(x, y) + Y_5(x, y))\hat{\boldsymbol{y}} = (x(y^2 - 1))\hat{\boldsymbol{x}} - (y(x^2 - 1))\hat{\boldsymbol{y}}. \tag{17}$$

From these explicit expressions we can easily calculate

$$\nabla \times \boldsymbol{E}_1 = 0, \quad \nabla \cdot \boldsymbol{E}_1 = x^2 + y^2 - 2 \tag{18}$$

$$\nabla \times \boldsymbol{E}_2 = -2y\hat{z}, \quad \nabla \cdot \boldsymbol{E}_2 = 0 \tag{19}$$

$$\nabla \times \boldsymbol{E}_3 = 2x\hat{z}, \quad \nabla \cdot \boldsymbol{E}_3 = 0 \tag{20}$$

$$\nabla \times \boldsymbol{E}_4 = -4xy\hat{z}, \quad \nabla \cdot \boldsymbol{E}_4 = y^2 - x^2. \tag{21}$$

We can also calculate the following integrals:

$$\int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{E}_j \, d\Omega = 0, \qquad i \neq j, \ i = 1, \cdots, 4, \ j = 1, \cdots, 4 \tag{22}$$

and

$$\int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot (\nabla \times \boldsymbol{E}_j) \, d\Omega = 0, \qquad i \neq j, \quad i = 1, \cdots, 4, \quad j = 1, \cdots 4. \tag{23}$$

From (22), it follows that $\{\boldsymbol{E}_k\}_{k=1}^4$ is an orthogonal basis of the finite-element space, i.e., a generic finite-element field $\boldsymbol{W}$ can be expanded as $\boldsymbol{W} = \Sigma_{k=1}^4 \alpha_k \boldsymbol{E}_k$.

In order to verify that the four finite-element vector fields $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$, $\boldsymbol{E}_4$ are just the eigenmodes of the considered discrete eigenproblem, let us calculate

$$\begin{aligned} &\int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot (\nabla \times \boldsymbol{W}) \, d\Omega \\ &= \int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot \left(\nabla \times \sum_{k=1}^4 \alpha_k \boldsymbol{E}_k \right) d\Omega \\ &= \sum_{k=1}^4 \alpha_k \int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot (\nabla \times \boldsymbol{E}_k) \, d\Omega \end{aligned} \tag{24}$$

and

$$\int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{W} \, d\Omega = \int_\Omega \boldsymbol{E}_i \cdot \sum_{k=1}^4 \alpha_k \boldsymbol{E}_k \, d\Omega = \sum_{k=1}^4 \alpha_k \int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{E}_k \, d\Omega. \tag{25}$$

By using (22) and (23), we obtain

$$\int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot (\nabla \times \boldsymbol{W}) \, d\Omega = \alpha_i \int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot (\nabla \times \boldsymbol{E}_i) \, d\Omega \tag{26}$$

and

$$\int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{W} \, d\Omega = \alpha_i \int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{E}_i \, d\Omega. \tag{27}$$

Now, let us define $k_i^2, i = 1, \cdots, 4$, by

$$\begin{aligned} &\int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot (\nabla \times \boldsymbol{E}_i) \, d\Omega \\ &= k_i^2 \int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{E}_i \, d\Omega, \qquad i = 1, \cdots, 4. \end{aligned} \tag{28}$$

Then, from (26) to (28), we have

$$\begin{aligned} &\int_\Omega (\nabla \times \boldsymbol{E}_i) \cdot (\nabla \times \boldsymbol{W}) \, d\Omega \\ &= k_i^2 \int_\Omega \boldsymbol{E}_i \cdot \boldsymbol{W} \, d\Omega, \quad \forall \boldsymbol{W} \in \text{span} \{\boldsymbol{E}_k\}_{k-1}^4, \qquad i = 1, \cdots, 4 \end{aligned} \tag{29}$$

and this means that $\boldsymbol{E}_i$ is an eigenvector and k_i^2 is the corresponding eigenvalue.

The inclusion condition [9] is

$$P_N(S) \subseteq S \tag{30}$$

where S is the space of vector fields spanned by the finite-element basis and $P_N(S)$ is the orthogonal projection of S on the space N of irrotational vector fields of the continuous problem (5). For the sake of precision

$$S = \text{span} \{\boldsymbol{E}_k\}_{k=1}^4 \tag{31}$$

and

$$N = \{\boldsymbol{v} \in H_0(\text{curl}, \Omega) | \nabla \times \boldsymbol{v} = 0\}. \tag{32}$$

Checking the inclusion condition directly is not an easy task. However, as proven in [10], the solenoidality of the nonirrotational modes of the discrete eigenproblem is necessary for the inclusion condition to be satisfied. However, $\boldsymbol{E}_4$ is just a mode of the discrete eigenproblem which is neither irrotational nor solenoidal (21). Hence, in the particular example considered, this necessary condition for the inclusion condition, and then the inclusion condition itself, are violated. This is sufficient to conclude that, in general, covariant projection elements do not satisfy the inclusion condition.

III. Conclusions

In [10] and [11] we have proven that the inclusion condition is sufficient, but not necessary to avoid spurious modes, and that edge elements do not satisfy it. It was still an open question whether covariant projection elements satisfy the inclusion condition or not—in this paper, we have proven that they do not.

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